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A VERSION OF THE ASYMPTOTIC THEORY OF THE UNSTEADY FREE INTERACTION OF A BOUNDARY LAYER WITH A TRANSONIC FLOW†

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The non-linear theory of strongly perturbed flows, with a structure of the velocity fields which is characteristic of domains of so-called free interaction of the boundary layer with an external potential flow, is considered. The specific details of the transonic flow show up not only in the estimates of the amplitudes and lengths of the perturbation waves in the asymptotic analysis of the problem but, also, in the fact that the motion may turn out to be simultaneously unsteady in the part of the boundary layer close to the wall and in the external potential flow. This mechanism of the evolution of the perturbations can be described by a single integro-differential equation which, assuming that the structure of the fluctuating fields is a quadrideck structure, is derived using the Fourier–Laplace method. Examples of its non-linear solutions are given in the form of solitary or periodic waves.

1. A BOUNDARY LAYER WITH A SELF-INDUCED PRESSURE IN TRANSONIC FLOW

A number of new aspects arise in Prandtl's theory when the concept of free interaction is introduced. According to this concept, unlike in the classical representations, a pressure gradient is induced by the boundary layer itself and it cannot be calculated from the solution of the external potential flow problem. A result of an asymptotic analysis of the system of Navier–Stokes equations [1–7] involves a derivation of Prandtl's equations with a self-induced pressure for the narrow subdomain which is in the immediate vicinity of the surface past which the flow occurs. This narrow subdomain is found to have a predominant effect on the growth of the displacement thickness of the boundary layer. Non-linear perturbation theory, which describes the process of free interaction, can be generalized to include unsteady flows [8–11], where the time derivatives in the equations of the first approximation only have to be retained in the above mentioned boundary sublayer if the free-stream velocity supersonic or subsonic. In the two other subdomains and, in fact, the main thickness of the boundary layer and the external potential flow, the gas motion is quasi-steady and the time derivatives only occur in the asymptotic equations for the higher approximations.

Conversely, in the case of transonic velocities of the gas motion, the dependence of the required functions on time turns out to be important precisely in the external potential part of the flow. This fact is an important feature in the extension of a previously proposed [12] asymptotic model of the interaction of a boundary layer with a transonic flow to unsteady motions [13]. Here, the velocity fields are found to be quasi-steady fields in the main thickness of the boundary layer and in the viscous sublayer. However, the generalization in [13] of the asymptotic scheme in [12] is not the only possible generalization. The alternative approach in [14] to the construction of a triple-deck theory of unsteady transonic flows gives rise, as a consequence, to a situation (which has not been previously encountered in asymptotic analysis) when terms containing time derivatives appear both in the system of equations for the viscous boundary sublayer and in the equation for the external potential perturbations (which, unlike the analogous equation in [13]), becomes linear).

If the amplitudes of the perturbations exceed the orders of magnitude which are dictated by the assumptions of the theory [1–7], the asymptotic analysis of the pulsation fields is based on a more complex structure of the flow field. In the case of the super- and subsonic ranges, this analysis leads to the formulation of a quadrideck asymptotic theory [15, 16], an important component of which is the proof of the applicability of the Burgers and Benjamin–Ono equations [17, 18] to the description of the evolution of perturbations.

The previously developed representations [15, 16] enable one to consider transonic flows with a quadrideck structure of the domain of interaction [19]. As previously [14], the wave pattern includes

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an extremely unsteady domain in the lower part of the boundary layer close to the wall and in the upper potential flow field. However, the asymptotic division of the domain of self-induced pressure into four sublayers is associated with the investigation of a class of perturbations which are characterized by a different normalization of the independent variables and the required functions, in particular, the greater relative magnitude of the amplitude compared with the normalization considered in [14]. The integro-differential equation, previously obtained in [19], which describes the free interaction process, reduces to the Burgers or Benjamin–Ono equations on leaving the transonic range (towards higher or lower Mach numbers). In this sense, the theory developed in [19] is an analogue of the approaches in [15, 16], which were proposed for Mach numbers of finite amplitude differing from unity.

Below, the main integro-differential equation in the asymptotic theory of the interaction of a boundary layer with a transonic flow, previously proposed in [19], is derived by another method, the Fourier–Laplace method.

2. A QUADRIDECK ASYMPTOTIC SCHEME OF FREE INTERACTION

Consider a flat plate, past which there a uniform flow of a compressible gas with a velocity U^*_∞, density ρ^*_∞, pressure p^*_∞, dynamic coefficient of viscosity μ^*_∞, speed of sound a^*_∞. Directing the x^* axis of a Cartesian system of coordinates (x^*, y^*) along the surface of the plate, we define the Reynolds number $\mathrm{Re} = \rho^*_\infty U^*_\infty L^*/\mu^*_\infty$ through a distance L^* from the leading edge (0, 0) up to a certain short domain in the neighbourhood of the point (L^*, 0). We shall denote dimensional quantities by an asterisk and specify an estimate of the smallness of the above-mentioned domain in terms of the Reynolds number $\mathrm{Re} \to \infty$ and a second parameter $\delta \to 0$ which characterizes the transonic nature of the flow as follows.

Suppose the Mach number $M_\infty = U^*_\infty/a^*_\infty$ of the free stream is close to unity, namely

$$M_\infty^2 = 1 + \delta Q_\infty, \quad Q_\infty = O(1), \quad \delta \to 0 \tag{2.1}$$

Everywhere, we will hence forth assume that the order of magnitude relation between the small parameter δ and the Reynolds number Re, which takes large values

$$\delta^3 \,\mathrm{Re} \to \infty \tag{2.2}$$

is satisfied.

Then, in the case of unsteady perturbations of the primarily steady flow (which includes a laminar boundary layer and uniform external flow), we introduce the dependence of the velocity components u^*, v^*, the density ρ^* and the pressure p^* on the time and the spatial coordinates by means of the fast variables

$$T = \delta^{5/2}\,\mathrm{Re}^{1/2}\frac{U^*_\infty t^*}{L^*}, \quad X = \delta^{3/2}\,\mathrm{Re}^{1/2}\frac{x^* - L^*}{L^*}, \quad Y_u = \delta^2\,\mathrm{Re}^{1/2}\frac{y^*}{L^*} \tag{2.3}$$

As can be seen from (2.3), the condition $X = O(1)$ fixes the longitudinal scale $\Delta x^* \sim \delta^{-3/2}\,\mathrm{Re}^{-1/2}L^*$ of the domain which is separated by a distance L^* from the leading edge and, moreover, according to (2.2), this scale is short: $\Delta x^* \ll L^*$. At the same time, the vertical scale $\Delta y^* \sim \delta^{-2}\,\mathrm{Re}^{-1/2}L^*$ of this domain, which is established by the condition $Y_u = O(1)$, is much greater than the thickness $\mathrm{Re}^{-1/2}L^*$ of the boundary layer which has been formed in the cross-section $x^* = L^*$. Hence, the normalization (2.3) of the vertical coordinate means that the boundary layer (deck 2 in Fig. 1) is located at $Y_u \to 0$ and, in the domain $Y_u = O(1)$ which is external to it (denoted as deck 1 in Fig. 1), viscosity and heat conduction are unimportant and the flow in it is vortex-free.

We write the equation for the potential $\Phi^* = \Phi^*(t^*, x^*, y^*)$ in deck *1* over the plane of the plate past which the flow occurs [20, 21] as

$$\frac{\partial^2\Phi^*}{\partial x^{*2}}\left[a^{*2} - \left(\frac{\partial\Phi^*}{\partial x^*}\right)^2\right] + \frac{\partial^2\Phi^*}{\partial y^{*2}}\left[a^{*2} - \left(\frac{\partial\Phi^*}{\partial y^*}\right)^2\right] - \frac{\partial^2\Phi^*}{\partial t^{*2}} -$$
$$-2\frac{\partial^2\Phi^*}{\partial x^*\partial y^*}\frac{\partial\Phi^*}{\partial x^*}\frac{\partial\Phi^*}{\partial y^*} - 2\frac{\partial^2\Phi^*}{\partial t^*\partial x^*}\frac{\partial\Phi^*}{\partial x^*} - 2\frac{\partial^2\Phi^*}{\partial t^*\partial y^*}\frac{\partial\Phi^*}{\partial y^*} = 0 \tag{2.4}$$

together with the Lagrange–Cauchy integral

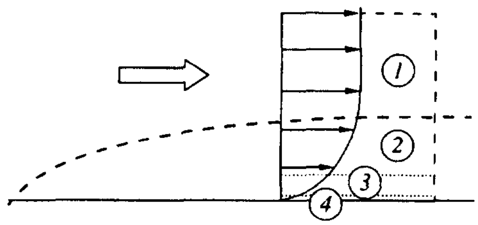

Fig. 1

$$\frac{\partial \Phi^*}{\partial t^*}+\frac{a^{*2}}{\gamma-1}+\frac{1}{2}\left[\left(\frac{\partial \Phi^*}{\partial x^*}\right)^2+\left(\frac{\partial \Phi^*}{\partial y^*}\right)^2\right]=\frac{a_\infty^{*2}}{\gamma-1}+\frac{U_\infty^{*2}}{2} \tag{2.5}$$

Here, γ is the ratio of the specific heat capacities. We represent the velocity potential

$$u^*=\frac{\partial \Phi^*}{\partial x^*}, \quad v^*=\frac{\partial \Phi^*}{\partial y^*} \tag{2.6}$$

in the form

$$\frac{\Phi^*}{L^* U_\infty^*}=1+\delta^{-3/2}\,\mathrm{Re}^{-1/2}\,X+\delta^{1/2}\,\mathrm{Re}^{-1/2}\,\varphi_{1u}(T,X,Y_u)+\ldots \tag{2.7}$$

The first two terms in (2.7) give the potential of the unperturbed flow $\Phi^* = U_\infty^* x^*$ at short distances $(x^* - L^*)L^{*-1} = O(\delta^{-3/2}\,\mathrm{Re}^{-1/2})$ for $X = O(1)$. Substituting (2.7) into (2.6) we obtain

$$\frac{u^*}{U_\infty^*}=1+\delta^2\frac{\partial \varphi_{1u}}{\partial X}+\ldots, \quad \frac{v^*}{U_\infty^*}=\delta^{5/2}\frac{\partial \varphi_{1u}}{\partial Y_u}+\ldots \tag{2.8}$$

Taking account of expansions (2.8), we write the Lagrange–Cauchy equation in the form

$$\frac{a^{*2}}{a_\infty^{*2}}=1-\delta^2 M_\infty^2(\gamma-1)\frac{\partial \varphi_{1u}}{\partial X}+\ldots \tag{2.9}$$

The density ρ^* and the pressure of the gas p^* are related by the equation of state and by the equation of a Poisson adiabatic curve which holds in the first approximation in the case of the inviscid domain considered here and are conveniently represented in the form

$$\frac{p^*-p_\infty^*}{\rho_\infty^* U_\infty^{*2}}=\frac{1}{\gamma M_\infty^2}\left[\left(\frac{\rho^*}{\rho_\infty^*}\right)^\gamma-1\right], \quad \frac{\rho^*}{\rho_\infty^*}=\left(\frac{a^*}{a_\infty^*}\right)^{2/(\gamma-1)} \tag{2.10}$$

Then, substituting the asymptotic expansion (2.9) into (2.10), we find

$$\frac{\rho^*}{\rho_\infty^*}=1-\delta^2\frac{\partial \varphi_{1u}}{\partial X}+\ldots, \quad \frac{p^*-p_\infty^*}{\rho_\infty^* U_\infty^{*2}}=-\delta^2\frac{\partial \varphi_{1u}}{\partial X}+\ldots \tag{2.11}$$

Hence, all the flow functions are expressed using formulae (2.8) and (2.11) in terms of the perturbation of the potential φ_{1u}. Note that relations (2.9)–(2.11) are in no way associated with assumption (2.1) regarding the closeness of the Mach number to unity, their derivation rests solely on estimate (2.2) for the two parameters $\delta \to 0$ and $\mathrm{Re} \to \infty$ occurring in (2.3) and (2.7).

As far as the function φ_{1u} is concerned, we obtain the equation for it by an asymptotic simplification of Eq. (2.4) subject to assumption (2.1) concerning the transonic nature of the flow using relations (2.3) and (2.7)–(2.9).

We substitute expressions (2.1), (2.7) and (2.9) into Eq. (2.4), and the leading terms will then be of order of magnitude $\tau_0 = \delta^{9/2}\,\mathrm{Re}^{1/2}\,U_\infty^{*3} L^{*-1}$. On discarding terms of higher infinitesimals $\sim \delta^{11/2}\,\mathrm{Re}^{1/2} U_\infty^{*3} L^{*-1}$ and cancelling out the factor τ_0 in the equation, we obtain

$$2\frac{\partial^2\varphi_{1u}}{\partial T\partial X}+Q_\infty\frac{\partial^2\varphi_{1u}}{\partial X^2}-\frac{\partial^2\varphi_{1u}}{\partial Y_u^2}=0 \tag{2.12}$$

Equation (2.12) was proposed in [14] within the framework of an alternative approach to the formulation of the theory of free interaction with the aim of retaining the terms determining the loss in stability of the boundary layer in a transonic external flow. Several other considerations were incorporated into the basis of the derivation of Eq. (2.12) in [19], where it is a condition for the compatibility of the asymptotic equations of the second approximation for perturbation of the gas dynamic functions in the upper deck of interaction.

Hence, the expression for the potential (2.7), in which the function $\varphi_{1u}(T, X, Y_u)$ satisfies Eq. (2.12), is equivalent to the following representations of the flow functions in deck *1* which is adjacent from above to the boundary layer

$$\frac{u^*}{U_\infty^*}=1+\delta^2 u_{1u}+\ldots,\quad \frac{v^*}{U_\infty^*}=\delta^{5/2}v_{1u}+\ldots$$
$$\frac{\rho^*}{\rho_\infty^*}=1+\delta^2\rho_{1u}\ldots,\quad \frac{p^*-p_\infty^*}{\rho_\infty^* U_\infty^{*2}}=\delta^2 p_{1u}+\ldots \tag{2.13}$$

where, in accordance with relations (2.8) and (2.11)

$$u_{1u}=\frac{\partial\varphi_{1u}}{\partial X},\quad v_{1u}=\frac{\partial\varphi_{1u}}{\partial Y_u},\quad p_{1u}=\rho_{1u}=-u_{1u} \tag{2.14}$$

The perturbations (2.13) and (2.14) penetrate into the boundary layer (deck *2*), initiating in it the fluctuation field

$$\frac{u^*}{U_\infty^*}=U_0+\delta u_{1m}+\delta^2 u_{2m}+\ldots,\quad \frac{v^*}{U_\infty^*}=\delta^{5/2}v_{1m}+\delta^{7/2}v_{2m}+\ldots$$
$$\frac{\rho^*}{\rho_\infty^*}=R_0+\delta\rho_{1m}+\delta^2\rho_{2m}+\ldots,\quad \frac{p^*-p_\infty^*}{\rho_\infty^* U_\infty^{*2}}=\delta^2 p_{1m}+\delta^3 p_{2m}+\ldots \tag{2.15}$$

Here $U_0 = U_0(Y_m), R_0 = R_0(Y_m)$ are the velocity and density profiles in the unperturbed steady-state laminar boundary layer, $Y_m = \mathrm{Re}^{1/2} L^{*-1} y^*$ is the vertical coordinate, which is of the order of unity in the interior 2 of the boundary layer and is related to the coordinate of the upper interaction deck *1*, which has been previously introduced by means of relations (2.3), as follows:

$$Y_m=\delta^{-2}Y_u \tag{2.16}$$

The variables T, X and Y_m are the arguments of the functions with subscripts $jm, j = 1, 2, \ldots$ in (2.15). Substituting expressions (2.15) into the Navier – Stokes equations we determine the first terms of the expansions, apart from the arbitrary functions $A_j(T, X)$

$$u_{1m}=A_1(T,X)\frac{dU_0}{dY_m},\quad v_{1m}=-\frac{\partial A_1}{\partial X}U_0(Y_m),\quad \rho_{1m}=A_1\frac{dR_0}{dY_m},\quad p_{1m}=p_{1m}(T,X)$$
$$u_{2m}=-U_0(Y_m)p_{1m}-\int_{-\infty}^{X}\frac{\partial v_{2m}}{\partial Y_m}dX' \tag{2.17}$$
$$v_{2m}=-U_0(Y_m)\frac{\partial p_{1m}}{\partial X}\int_{Y_m}^{\infty}\left(\frac{1}{R_0U_0^2}-1\right)dY_m'-\frac{\partial A_1}{\partial T}-A_1\frac{\partial A_1}{\partial X}\frac{dU_0}{dY_m}-\frac{\partial A_2}{\partial X}U_0(Y_m)$$

The behaviour of the solution of (2.17) in the lower part of the boundary layer follows from the limiting properties

$$U_0=\lambda_1 Y_m+\ldots,\quad R_0=r_0+\ldots\quad (Y_m\to 0) \tag{2.18}$$

Taking account of (2.18), we write the internal limit of the expansions (2.15) and (2.17) when $Y_m \to 0$ in the form

$$\frac{u^*}{U_\infty^*} \to \lambda_1(Y_m + \delta A_1) + \dots, \quad \frac{\rho^*}{\rho_\infty^*} \to r_0 + \dots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} \to \delta^2 p_{1m} + \dots \tag{2.19}$$

$$\frac{v^*}{U_\infty^*} \to -\delta^{5/2}\lambda_1 \frac{\partial A_1}{\partial X} Y_m - \delta^{7/2}\left(\frac{\partial A_1}{\partial T} + \lambda_1 A_1 \frac{\partial A_1}{\partial X} + \frac{1}{\lambda_1 r_0}\frac{\partial p_{1m}}{\partial X}\right) + \dots$$

If $Y_m = O(\delta)$, then, by relations (2.19), the velocity perturbation $\delta\lambda_1 A_1$ becomes a quantity of the order of magnitude of the unperturbed velocity $\lambda_1 Y_m$. Consequently, a non-linear deck *3* (Fig. 1) arises in the lower part of the boundary layer *2*. We define the variable Y_a by the formula

$$Y_a = \delta^{-1} Y_m \tag{2.20}$$

and then $Y_a = O(1)$ in deck *3*.

Relations (2.14) and (2.17) were obtained without taking account of the viscous terms in the Navier–Stokes equations. In the case of perturbations of the form of (2.15), viscous effects manifest themselves in the thin sublayer (deck *4* in Fig. 1) which comes into contact with the wall. This sublayer has a thickness of the order of $Y_m = O(\beta_0)$, where the small parameter

$$\beta_0 = \delta^{-5/4} \mathrm{Re}^{-1/4} \tag{2.21}$$

We now introduce a new vertical coordinate

$$Y_l = \beta_0^{-1} Y_m \tag{2.22}$$

Then, in the above mentioned viscous layer, the perturbations of the flow functions depend on the variables T, X and Y_l and they are given by the expansions

$$\frac{u^*}{U_\infty^*} = \delta u_{1l} + \dots, \quad \frac{v^*}{U_\infty^*} = \delta^{5/4} \mathrm{Re}^{-1/4} v_{1l} + \dots \tag{2.23}$$

$$\frac{\rho^*}{\rho_\infty^*} = r_0 + \dots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \delta^2 p_{1l} + \dots$$

We will now consider the question of the ratio of the thicknesses of decks *3* and *4*, which depends on the asymptotic estimates of the parameter δ in terms of the Reynolds number Re. Relation (2.2), as was noted above, means that the longitudinal scale Δx^* is small compared with L^*.

We assume that $Y_a = Y_l$ which is equivalent to the assertion that decks *3* and *4* are superposed. Comparison of the right-hand sides of equalities (2.20) and (2.22) gives

$$\delta = \mathrm{Re}^{-1/9} \tag{2.24}$$

Relation (2.24) guarantees that condition (2.2) is satisfied. The case of (2.24), when the domain of disturbed motion has a three-layer structure and the non-linear and viscous sublayers merge with one another, has been considered previously in [14].

We shall reinforce the constraint (2.2) even further and require that the condition

$$\delta^9 \mathrm{Re} \to \infty \tag{2.25}$$

must be satisfied instead of (2.24).

By virtue of the dependence

$$Y_l = \delta^{9/4} \mathrm{Re}^{1/4} Y_a \tag{2.26}$$

which follows from relations (2.20) and (2.22), it can be concluded that $Y_a \to 0$ when $Y_l = O(1)$ in the case of (2.25). This means that the thickness of deck *4*, which is adjacent to the wall, is much less than the thickness of deck *3*. In other words, if the amplitude of the perturbations δ (in expansions (2.13), (2.15) and (2.23)) exceeds the value of (2.24) in order of magnitude and condition (2.25) is satisfied, then the flow field acquires a quadrideck structure [19] where the viscous sublayer *4* is located at the bottom of the non-linear sublayer *3* (Fig. 1).

It remains to point out that the right-hand sides of the limiting expressions (2.19), rewritten in terms of the variable Y_a, serve as exact solutions of the asymptotic equations for the inviscid non-linear deck *3*. Hence, expansions (2.15) and (2.17) and their inner limit (2.19) hold up to the upper boundary of the viscous sublayer *4*, that is, when $Y_m = O(\beta_0)$, $Y_a = O(\beta_0\delta^{-1})$, and they must be matched with the functions (2.23). However, it follows from condition (2.25) that $\beta_0 \ll \delta$ and it is clear from relation (2.26) that the upper boundary of the viscous sublayer corresponds to $Y_m\delta^{-1} = Y_a \to 0$. Hence, even without matching (2.19) to (2.23), it can be shown directly that the necessary condition for its realizability is just

$$\frac{\partial A_1}{\partial T} + \lambda_1 A_1 \frac{\partial A_1}{\partial X} = -\frac{1}{\lambda_1 r_0}\frac{\partial p_{1m}}{\partial X} \tag{2.27}$$

We now consider the outer limit as $Y_m \to \infty$ of expansions (2.15) and (2.17) which, by relation (2.16), must be equal to the inner limit, as $Y_u \to 0$, of expansions (2.8) and (2.11) for the upper flow deck *1*. Since

$$U_0(Y_m) \to 1, \quad R_0(Y_m) \to 1 \quad (Y_m \to \infty)$$

the asymptotic sequences (2.15) at the upper edge of boundary layer 2 are transformed in the following manner:

$$\begin{aligned} &\frac{u^*}{U_\infty^*} \to 1 - \delta^2 p_{1m} + \ldots, \quad \frac{v^*}{U_\infty^*} \to -\delta^{5/2}\frac{\partial A_1}{\partial X} + \ldots \\ &\frac{\rho^*}{\rho_\infty^*} \to 1 + \delta^2 p_{1m} \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} \to \delta^2 p_{1m} + \ldots \end{aligned} \tag{2.28}$$

The outer asymptotic form (2.28) establishes the limiting form of expressions (2.8) and (2.11), taken as $Y_u \to 0$, which gives

$$\frac{\partial \varphi_{1u}(T,X,0)}{\partial X} = -p_{1m}(T,X), \quad \frac{\partial \varphi_{1u}(T,X,0)}{\partial Y_u} = -\frac{\partial A_1(T,X)}{\partial X} \tag{2.29}$$

We now eliminate the unimportant constants λ_1, r_0 from relations (2.12) (2.27) and (2.29) by a change in the scales of the quantities using the previously indicated similarity transformation [19]. Changing from $T, X, Y_u, A_1, p_{1m}, \varphi_{1u}, Q_\infty$ to the variables $t, x, y_u, A, p, \varphi, K_\infty$ respectively, we write the closed system of inviscid equations in the final form

$$\frac{\partial^2 \varphi}{\partial t \partial x} + K_\infty \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y_u^2} = 0 \tag{2.30}$$

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = -\frac{\partial p}{\partial x} \tag{2.31}$$

$$\frac{\partial \varphi(t,x,0)}{\partial x} = -p(t,x), \quad \frac{\partial \varphi(t,x,0)}{\partial y_u} = -\frac{\partial A(t,x)}{\partial x} \tag{2.32}$$

We will now show that the system of equations (2.30)–(2.32) yields a single integro-differential equation for the function $A(t,x)$, and, when deriving this equation, we shall be guided by several other arguments compared with those previously used in [19].

3. INTEGRAL FORMULA FOR THE POTENTIAL ESTABLISHED BY THE FOURIER–LAPLACE METHOD

In the equation for the potential (2.30), we carry out a Fourier transformation with respect to the variable x

$$\varphi^*(t,k,y_u)=\int_{-\infty}^{\infty}\varphi(t,x,y_u)e^{-ikx}dx,\quad \varphi(t,x,y_u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi^*(t,k,y_u)e^{ikx}dk$$

and a Laplace transformation with respect to the variable t

$$\varphi^{**}(\lambda,k,y_u)=\int_{0}^{\infty}\varphi^*(t,k,y_u)e^{-\lambda t}dt,\quad \varphi^*(t,k,y_u)=\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty}\varphi^{**}(\lambda,k,y_u)e^{\lambda t}d\lambda$$

We shall assume that $\varphi(t,x,y_u)=0$, $t\leqslant 0$. Then

$$-\frac{\partial^2\varphi^{**}}{\partial y_u^2}+ik\lambda\varphi^{**}-k^2K_\infty\varphi^{**}=0 \tag{3.1}$$

We put $i=e^{i\pi/2}$, assuming

$$(ik)^{1/2}=\frac{1\pm i}{\sqrt{2}}|k|^{1/2}=i^{\pm 1/2}|k|^{1/2}$$

in the expressions containing multivalued functions, where the upper sign is taken when $k>0$ and the lower sign when $k<0$. The solution of Eq. (2.1), which decays as $y_u\to+\infty$, has the form

$$\varphi^{**}(\lambda,k,y_u)=\varphi^{**}(\lambda,k,0)\exp\left[-(ik)^{1/2}(\lambda+ikK_\infty)^{1/2}y_u\right] \tag{3.2}$$

In order to determine the quantity $\varphi^{**}(\lambda,k,0)$, we make use of the second boundary condition from (2.32) which has been rewritten in terms of the Fourier–Laplace transforms

$$\frac{\partial\varphi^{**}(\lambda,k,0)}{\partial y_u}=-ikA^{**}(\lambda,k) \tag{3.3}$$

Differentiating equality (3.2) with respect to y_u, from condition (3.3) we find

$$\varphi^{**}(\lambda,k,0)=\frac{(ik)^{1/2}A^{**}(\lambda,k)}{(\lambda+ikK_\infty)^{1/2}} \tag{3.4}$$

Hence

$$\varphi^{**}(\lambda,k,y_u)=(ik)^{1/2}A^{**}(\lambda,k)\Lambda^{**}(\lambda,k,y_u)=\left[\frac{\partial A}{\partial x}\right]^{**}(\lambda,k)\frac{\Lambda^{**}(\lambda,k,y_u)}{(ik)^{1/2}} \tag{3.5}$$

where

$$\Lambda^{**}(\lambda,k,y_u)=\frac{\exp\left[-(ik)^{1/2}(\lambda+ikK_\infty)^{1/2}y_u\right]}{(\lambda+ikK_\infty)^{1/2}}$$

We use the formula [23]

$$\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty}e^{-\alpha\sqrt{\lambda}+\lambda t}\frac{d\lambda}{\sqrt{\lambda}}=\frac{1}{\sqrt{\pi t}}e^{-\alpha^2/(4t)} \tag{3.6}$$

which has been proved using the Poisson integral [22]

$$\int_0^\infty e^{-ts^2}\cos\alpha s\,ds=\frac{1}{2}\sqrt{\frac{\pi}{t}}e^{-\alpha^2/(4t)}\quad(t>0)$$

and calculate the inverse Laplace transform for the function $\Lambda^{**}(\lambda, k, y_u)$. Making the substitution $\lambda = \lambda_* - ikK_\infty$ in transform formula (3.6), we find

$$\Lambda^*(t,k,y_u)=\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty}\Lambda^{**}(\lambda,k,y_u)e^{\lambda t}d\lambda=$$

$$=\frac{1}{2\pi i}e^{-ikK_\infty t}\int_{l-i\infty}^{l+i\infty}e^{-(ik)^{1/2}\lambda_*^{1/2}y_u+\lambda_* t}\frac{d\lambda_*}{\lambda_*^{1/2}}=\frac{1}{\sqrt{\pi t}}e^{-ikK_\infty t-iky_u^2/(4t)}\tag{3.7}$$

According to the multiplication theorem [23], the inverse transform of the product of the transforms is expressed in terms of a convolution of the inverse transforms of the factors

$$\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty}\Lambda^{**}(\lambda,k,y_u)A^{**}(\lambda,k)e^{\lambda t}d\lambda=\int_0^t\Lambda^*(t-\tau,k,y_u)A^*(\tau,k)d\tau$$

The inverse transform (3.7) of the function $\Lambda^{**}(\lambda, k, y_u)$ which has been found above therefore enables one to invert the Laplace transform $\varphi^{**}(\lambda, k, y_u)$ from (3.5)

$$\varphi^*(t,k,y_u)=\frac{1}{(ik)^{1/2}}\int_0^t\left[\frac{\partial A}{\partial x}\right]^*(\tau,k)e^{-ikK_\infty(t-\tau)-iky_u^2/[4(t-\tau)]}\frac{d\tau}{\sqrt{\pi(t-\tau)}}\tag{3.8}$$

We now put $y_u = 0$ in (3.8) and calculate the inverse Fourier transform of the function $\varphi^*(t, k, 0)$ with respect to the variable x

$$\varphi(t,k,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}dk}{(ik)^{1/2}}\int_0^t e^{-ikK_\infty(t-\tau)}\frac{d\tau}{\sqrt{\pi(t-\tau)}}\int_{-\infty}^{\infty}\frac{\partial A(\tau,\xi)}{\partial\xi}e^{-ik\xi}d\xi=$$

$$=\frac{1}{2\pi}\int_0^t\frac{d\tau}{\sqrt{\pi(t-\tau)}}\int_{-\infty}^{\infty}\frac{\partial A(\tau,\xi)}{\partial\xi}d\xi\left[\int_{-\infty}^{0}\frac{e^{-ik(\xi-Q)}}{(-i|k|)^{1/2}}dk+\int_0^\infty\frac{e^{-ik(\xi-Q)}}{(ik)^{1/2}}dk\right]\tag{3.9}$$

Here, $Q = x - K_\infty(t - \tau)$. Internal integration with respect to k in relation (3.9) gives

$$\int_0^\infty\frac{e^{i|k|(\xi-Q)}}{(-i|k|)^{1/2}}d|k|+\int_0^\infty\frac{e^{-ik(\xi-Q)}}{(ik)^{1/2}}dk=2\,\mathrm{Im}\int_0^\infty\frac{i^{1/2}e^{-ik(\xi-Q)}}{k^{1/2}}dk=$$

$$=2\,\mathrm{Im}\frac{i^{1/2}}{\sqrt{|\xi-Q|}}\int_0^\infty\frac{e^{-i\omega\,\mathrm{sign}(\xi-Q)}}{\omega^{1/2}}d\omega=\begin{cases}0, & \xi>Q\\ \dfrac{2\sqrt{\pi}}{\sqrt{Q-\xi}}, & \xi<Q\end{cases}$$

since [23]

$$\int_0^\infty\frac{e^{\pm i\omega}}{\omega^{1/2}}d\omega=2(\pm i)^{1/2}\int_0^{\infty(\pm i)^{-1/2}}e^{-\gamma^2}d\gamma=(\pm i)^{1/2}\sqrt{\pi}$$

Hence, the domain of integration S in relation (3.9) with respect to the variable τ and ξ is given by the inequalities

$$S:\ 0<\tau<t,\quad \xi<Q=x-K_\infty(t-\tau)$$

Consequently, inversion of the Fourier–Laplace integral transforms under the assumption that

$\varphi(0, x, y_u) = 0$ establishes the relation

$$\varphi(t,x,0) = \frac{1}{\pi}\int_0^t \frac{d\tau}{\sqrt{t-\tau}} \int_{-\infty}^{Q} \frac{\partial A(\tau,\xi)}{\partial \xi} \frac{d\xi}{\sqrt{Q-\xi}} =$$

$$= \frac{1}{\pi}\int_{-\infty}^{x-K_\infty t} \frac{d\xi^\circ}{\sqrt{x-K_\infty t-\xi^\circ}} \int_0^t \frac{\partial A(\tau,\xi^\circ+K_\infty\tau)}{\partial \xi^\circ} \frac{d\tau}{\sqrt{t-\tau}} = B(t,x;0) \tag{3.10}$$

$$B(t,x;a) = \frac{1}{\pi}\int_a^t d\tau \int_{-\infty}^{x-K_\infty(t-\tau)} \frac{\partial A(\tau,\xi)}{\partial \xi} \frac{d\xi}{\sqrt{(t-\tau)(x-\xi)-K_\infty(t-\tau)^2}}$$

Now, suppose the initial data are non-trivial:

$$\varphi(0,x,y_u) = \varphi_0(x,y_u)$$

On applying a Fourier–Laplace transformation with respect to the variables t and x to the equation for the potential once again, we obtain the equation

$$-\frac{\partial^2\varphi^{**}}{\partial y_u^2} + ik\lambda\varphi^{**} - k^2K_\infty\varphi^{**} = f_0 \tag{3.11}$$

in $\varphi^{**} = \varphi^{**}(\lambda, k, y_u)$, which differs from (3.1) in the fact that there is an inhomogeneous term $f_0 = ik\varphi_0^*(k, y_u) = ik\varphi^*(0, k, y_u)$. In order to take account of this term, we initially find the solution of Eq. (3.11) with the homogeneous boundary conditions

$$y_u = 0:\ \ \partial\varphi^{**}/\partial y_u = 0, \quad y_u \to +\infty:\ \ \varphi^{**} \to 0 \tag{3.12}$$

We now construct Green's function of boundary-value problem (3.11), (3.12) on the semi-axis $y_u \geqslant 0$. For brevity, we use the notation

$$\vartheta = (ik)^{1/2}(\lambda + ikK_\infty)^{1/2} \tag{3.13}$$

and consider the two linearly independent solutions of the homogeneous equation (3.1)

$$\psi_1(y_u) = e^{\vartheta y_u} + e^{-\vartheta y_u}, \quad \psi_2(y_u) = e^{-\vartheta y_u}$$

with the Wronskian $w = \psi_2'\psi_1 - \psi_2\psi_1' = -2\vartheta$. The function ψ_1 satisfies the first boundary condition of (3.2) and, by virtue of the fact that $\arg\vartheta \in (-\pi/2, \pi/2)$ for any sing of k, the function ψ_2 satisfies boundary condition (3.12)

Green's function [24]

$$G(y_u, y_u') = -\frac{1}{w}\begin{cases}\psi_1(y_u)\psi_2(y_u'), & 0 \leqslant y_u \leqslant y_u' \\ \psi_2(y_u)\psi_1(y_u'), & 0 \leqslant y_u' \leqslant y_u\end{cases}$$

yields the solution of boundary-value problem (3.11), (3.12) in the form

$$\varphi^{**}(\lambda,k,y_u) = \int_0^\infty G(y_u,y_u')f_0(y_u')dy_u' =$$

$$= -\frac{1}{w}\left[\psi_2(y_u)\int_0^{y_u}\psi_1(y_u')f_0(y_u')dy_u' + \psi_1(y_u)\int_{y_u}^\infty \psi_2(y_u')f_0(y_u')dy_u'\right] =$$

$$= \frac{1}{2\vartheta}[e^{-\vartheta y_u}F_+(\lambda,k;0,y_u) + e^{\vartheta y_u}F_-(\lambda,k;y_u,\infty) + e^{-\vartheta y_u}F_-(\lambda,k;0,\infty)] \tag{3.14}$$

$$F_\pm(\lambda,k;a,b) = \int_a^b e^{\pm\vartheta y_u'} f_0(y_u')dy_u'$$

In particular, when $y_u = 0$, we have, from relation (3.14)

$$\chi^{**}(\lambda,k)=\varphi^{**}(\lambda,k,0)=\frac{1}{\vartheta}F_{-}(\lambda,k;0,\infty) \tag{3.15}$$

The required solution of the initial-boundary-value problem for Eq. (2.30) is obtained as the sum of the solutions of problems with zero Cauchy data and with a homogeneous boundary condition when $y_u = 0$. Their transforms in spectral space after the Fourier–Laplace transformation are given by expressions (3.5) and (3.14); inversion of these transforms yields the potential motion pattern in the half-plane $y_u \geqslant 0$, if the boundary function $\partial\varphi(t,x,0)/\partial y_u$ is known.

However, the multistructured asymptotic theory considered here is characterized by the fact that the velocity potential $\varphi(t,x,y_u)$ can only be found as a result of the simultaneous solution of the boundary-value problems within the boundary layer and the outer deck described by Eq. (2.30), which is adjacent to it from above. Hence, neither the potential function $\varphi(t,x,0)$ nor its normal derivative $\partial\varphi(t,x,0)/\partial y_u$ are known in advance.

On the other hand, these two functions cannot be specified independently; we obtain the relation between them in spectral space by summing expressions (3.5) (when $y_u = 0$ and taking account of condition (3.3)) and (3.15), that is,

$$\varphi^{**}(\lambda,k,0)=-\frac{1}{\vartheta}\frac{\partial\varphi^{**}(\lambda,k,0)}{\partial y_u}+\frac{ik}{\vartheta}\int_0^\infty e^{-\vartheta y_u'}\varphi^{*}(0,k,y_u')dy_u' \tag{3.16}$$

where the quantity ϑ is defined in (3.13). Since the inverse Laplace transformation of the first term in relation (3.16) has already been calculated and is expressed by equality (3.8) (when $y_u = 0$), we shall find the original of the second term. Again, by using formula (3.6) and and making the same change of the variable of integration as in (3.7), we obtain

$$\chi^{*}(t,k)=\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty}e^{\lambda t}d\lambda\int_0^\infty\frac{(ik)^{1/2}e^{-(ik)^{1/2}(\lambda+ikK_\infty)^{1/2}y_u'}}{(\lambda+ikK_\infty)^{1/2}}\varphi^{*}(0,k,y_u')dy_u'=$$

$$=(ik)^{1/2}\int_0^\infty\varphi^{*}(0,k,y_u')e^{-ikK_\infty t}\left[\frac{1}{2\pi i}\int_{l-i\infty}^{l+i\infty}e^{-(ik)^{1/2}\lambda_*^{1/2}y_u'+\lambda_* t}\frac{d\lambda_*}{\lambda_*^{1/2}}\right]dy_u'=I_0(t,k)$$

$$I_a(t,k)=(ik)^{1/2}\int_0^\infty\varphi^{*}(a,k,y_u')e^{-ikK_\infty t-iky_u'^2/(4t)}\frac{dy_u'}{\sqrt{\pi t}}$$

for $\chi^*(t,k) = \varphi^*(t,k,0)$ from equality (3.15).

Hence, by using inversion of the Laplace transform, an integral relation is obtained between the Fourier transforms of the potential and its normal derivative (which is given by the second relation of (2.32)) on the boundary $y_u = 0$

$$\varphi^{*}(t,k,0)=\Phi(t,k;0)+I_0(t,k) \tag{3.17}$$

$$\Phi(t,k;a)=\frac{1}{(ik)^{1/2}}\int_a^t\left[\frac{\partial A(\tau,x)}{\partial x}\right]^{*}e^{-ikK_\infty(t-\tau)}\frac{d\tau}{\sqrt{\pi(t-\tau)}}$$

We put $t = t' - t_0$ on the right-hand and left-hand sides of equality (3.17), thereby changing to the functions

$$A'(t',x)=A(t'-t_0,x)=A(t,x),\quad \varphi'(t',x,y_u)=\varphi(t'-t_0,x,y_u)=\varphi(t,x,y_u)$$

We next replace the variable of integration: $\tau = \tau' - t_0$ in the first term on the right-hand side of equality (3.17). Then, instead of (3.17), we obtain a more general formula (the primes on the new functions and independent variables are omitted)

$$\varphi^{*}(t,k,0)=\Phi(t,k;t_0)+I_{t_0}(t-t_0,k) \tag{3.18}$$

The first integral from equality (3.18) when $t_0 = -\infty$, which is understood as being an improper integral, converges (according to the Dirichlet test [22]). Assuming the absolute convergence of the improper integral

$$\int_0^\infty |\varphi^*(t_0, k, y'_u)|\, dy'_u = C^*(t_0, k)$$

we now estimate the second integral from equality (3.18) when $t_0 \to -\infty$

$$|I_{t_0}(t - t_0, k)| \leq \sqrt{\frac{|k|}{\pi(t - t_0)}} \int_0^\infty |\varphi^*(t_0, k, y'_u)|\, dy'_u =$$

$$= \frac{|k|^{1/2}}{|t_0|^{1/2}\sqrt{\pi}}\left[1 - \frac{t}{2|t_0|} + O\left(\frac{t^2}{t_0^2}\right)\right] C^*(t_0, k) \qquad (3.19)$$

Taking the limit as $t_0 \to -\infty$ we obtain (in the case when $C^\infty(t_0, k)$ is bounded with respect to t_0) the special case of the representation of (3.18) for the Fourier transform of the potential function on the boundary $y_u = 0$

$$\varphi^*(t, k, 0) = \Phi(t, k; -\infty) \qquad (3.20)$$

The estimate (3.19) serves as a basis for the possible replacement, in (3.10), of the lower limit of integration with respect to the time variable τ by $-\infty$. On repeating the calculations carried out when calculating the inverse Fourier transformation and when deriving expression (3.1) from (3.8), we obtain

$$\varphi(t, x, 0) = B(t, x; -\infty) \qquad (3.21)$$

Integral relation (3.21) was found earlier in [19] by somewhat different arguments, namely, by starting out from representations of the solutions of the wave equation, which are known in the theory of potential flows [25], in terms of the values of the normal derivative on the boundary. If the Cauchy data are given for $t = t_0$, then t_0 has to be taken as the lower limit of integration in equality (3.21) and a term added which is the inverse fourier transformation of the second integral term in equality (3.18).

Substituting expression (3.21) into the first of relations (2.32) we can eliminate the function p from Eq. (2.31). Equation (2.31) is thereby transformed into an integro-differential equation in the function A and can be written in the form

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{1}{\pi}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{t} d\tau \int_{-\infty}^{x - K_\infty(t-\tau)} \frac{\partial A(\tau, \xi)}{\partial \xi} \frac{d\xi}{\sqrt{(t-\tau)(x-\xi) - K_\infty(t-\tau)^2}} \qquad (3.22)$$

4. LIMITING CASES

We will now consider the asymptotic form of expression (3.20) when $K_\infty \to +\infty$ and change from the variable of integration $\tau \in (-\infty, t)$ to the new variable $\nu \in (0, +\infty)$ using the formula

$$\tau = t - \frac{\nu^2}{|k| K_\infty}$$

Taking into account the fact that

$$\frac{\partial A(\tau, x)}{\partial x} = \frac{\partial}{\partial x} A\left(t - \frac{\nu^2}{|k| K_\infty}, x\right) = \frac{\partial A(t, x)}{\partial x} + O\left(\frac{\nu^2}{|k| K_\infty}\right) \qquad (4.1)$$

and using the value of the Fresnel integral [22]

$$\int_0^\infty e^{\mp i\eta^2} d\eta = (\mp i)^{1/2} \frac{\sqrt{\pi}}{2}$$

with an accuracy $O(|k|^{-3/2}K_\infty^{-3/2})$, we have

$$\varphi^*(t,k,0) = \mp\frac{i}{|k|\sqrt{K_\infty}}\left[\frac{\partial A(t,x)}{\partial x}\right]^* = \frac{A^*(t,k)}{\sqrt{K_\infty}}, \quad \varphi(t,x,0) = \frac{A(t,x)}{\sqrt{K_\infty}} \tag{4.2}$$

Here, as everywhere, the upper sign in the symbols $\pm$ or $\mp$ corresponds to $k > 0$ and the lower sign corresponds to $k < 0$.

By virtue of relations (2.31) and (2.32), Eqs (4.2) reduce to Burgers equation

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{1}{\sqrt{K_\infty}}\frac{\partial^2 A}{\partial x^2} \tag{4.3}$$

Hence, Eq. (4.3) is the limiting form of Eq. (3.22) when $K_\infty \to +\infty$.

Now suppose that $K_\infty \to -\infty$. On introducing the new variable of integration

$$\tau = t + \frac{v^2}{|k|K_\infty}$$

from relation (3.20), as previously, we find

$$\varphi^*(t,k,0) = \frac{1}{|k|\sqrt{-K_\infty}}\left[\frac{\partial A(t,x)}{\partial x}\right]^* = i\frac{A^*(t,k)}{\sqrt{-K_\infty}}\operatorname{sign} k \tag{4.4}$$

The well-known property of the Hilbert operator (which is proved, for example, using the Sokhotskii formulae [23])

$$\frac{1}{\pi}\text{v.p.}\int_{-\infty}^{\infty} e^{ik\xi}\frac{d\xi}{\xi - x} = i\operatorname{sign} k e^{ikx} \tag{4.5}$$

enables us to change in (4.4) from the Fourier transforms to their inverse transforms:

$$\varphi(t,x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi^*(t,k,0)e^{ikx}dk = \frac{1}{\pi}\text{v.p.}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A^*(t,k)}{\sqrt{-K_\infty}}e^{ik\xi}dk\right]\frac{d\xi}{\xi - x}$$

Consequently, relation (4.4) establishes the relation

$$\varphi(t,x,0) = \frac{1}{\pi\sqrt{-K_\infty}}\text{v.p.}\int_{-\infty}^{\infty}\frac{A(t,\xi)}{\xi - x}d\xi \tag{4.6}$$

but relations (2.31) and (2.32) then reduce to the Benjamin–Ono equations [17, 18]

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{1}{\pi\sqrt{-K_\infty}}\text{v.p.}\int_{-\infty}^{\infty}\frac{\partial^2 A(t,\xi)}{\partial \xi^2}\frac{d\xi}{\xi - x} \tag{4.7}$$

Hence, we have Eq. (4.7) as the limiting form of Eq. (3.22) as $K_\infty \to -\infty$

5. A TRAVELLING-WAVE TYPE SELF-SIMILARITY

We will now consider a class of solutions of system (2.30)–(2.32) in which the dependence of all the functions on the variables t and x is specified in a self-similar form. In particular

$$\varphi(t,x,y_u) = \tilde{\varphi}(x - ct, y_u)$$

In this case, we have from Eq. (2.30)

$$(c - K_\infty)\frac{\partial^2\tilde{\varphi}}{\partial x^2} + \frac{\partial^2\tilde{\varphi}}{\partial y_u^2} = 0 \tag{5.1}$$

Suppose $K_\infty > c$. Denoting two arbitrary functions by $g(x)$ and $h(x)$, we write the general solution of Eq. (5.1) in the form

$$\tilde{\varphi} = g(\zeta - \eta) + h(\zeta + \eta); \qquad \zeta = x - ct, \quad \eta = y_u\sqrt{K_\infty - c} \tag{5.2}$$

The requirement that $\tilde{\varphi} \to 0$ as $x \to -\infty$ uniformly with respect to y_u excludes the propagation of perturbations along the characteristics of the second family $\zeta + \eta$ = const. and, consequently, it is necessary to put $h \equiv 0$ in (5.2). Perturbations from the main thickness of the boundary layer are transmitted to its upper edge and are carried away downstream along the characteristics of the first family $\zeta - \eta$ = const. Whatever the function $g(x)$, substitution of expression (5.2) with $h \equiv 0$ into the two conditions (2.32) establishes the differential relation between $p = \tilde{p}(x - ct)$ and $A = A(x - ct)$

$$\tilde{p} = -\frac{1}{\sqrt{K_\infty - c}}\frac{\partial \tilde{A}}{\partial x} \tag{5.3}$$

From relations (2.31), (2.32) and (5.3), we have Burgers equation

$$(\tilde{A} - c)\frac{\partial \tilde{A}}{\partial x} = \frac{1}{\sqrt{K_\infty - c}}\frac{\partial^2 \tilde{A}}{\partial x^2} \tag{5.4}$$

Let us now assume that $K_\infty < c$. Equation (5.1) is Laplace's equation in the variables ζ and η. The solution of Neuman's problem for Laplace's equation in the upper half-plane $\eta > 0$ is written in the form [24]

$$\tilde{\varphi}(\zeta, \eta) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial \tilde{\varphi}(\xi, 0)}{\partial \eta}\ln[(\zeta - \xi)^2 + \eta^2]d\xi \tag{5.5}$$

Differentiation of Eq. (5.5) with respect to ζ leads to one of the forms of Poisson's formula [23]

$$\frac{\partial \tilde{\varphi}(\zeta, \eta)}{\partial \zeta} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial \tilde{\varphi}(\xi, 0)}{\partial \eta}\frac{\zeta - \xi}{(\zeta - \xi)^2 + \eta^2}d\xi \tag{5.6}$$

We now introduce the variable of integration $s = \xi - \zeta$ and, for any $\varepsilon > 0$, subdivide the integral on the right-hand side of (5.6) into three integrals, separating an integral in the section $[-\varepsilon, \varepsilon]$, and take the limit on both sides of Eq. (5.6) as $\eta \to +0$ and then as $\varepsilon \to +0$. As a result, we obtain that the following relation holds on the boundary $y_u = 0$

$$\frac{\partial \tilde{\varphi}(\zeta, 0)}{\partial \zeta} = \frac{1}{\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{\partial \tilde{\varphi}(s + \zeta, 0)}{\partial \eta}\frac{ds}{s} = \frac{1}{\pi\sqrt{c - K_\infty}}\,\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{\partial \tilde{\varphi}(\xi, 0)}{\partial y_u}\frac{d\xi}{\xi - \zeta} \tag{5.7}$$

The two conditions (2.32)

$$p = \tilde{p}(\zeta) = -\frac{\partial \tilde{\varphi}(\zeta, 0)}{\partial \zeta}, \qquad \frac{\partial A}{\partial x} = \frac{\partial \tilde{A}(\zeta)}{\partial \zeta} = -\frac{\partial \tilde{\varphi}(\xi, 0)}{\partial \eta}$$

together with relation (5.9) establish the integral relation

$$\tilde{p} = -\frac{1}{\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{\partial \tilde{A}}{\partial \xi}\frac{d\xi}{\xi - \zeta} \tag{5.8}$$

Together with $\tilde{\varphi}(\zeta, \eta)$, the function $\partial\tilde{\varphi}(\zeta, \eta)/\partial\zeta$ also satisfies Laplace's equation and, on repeating the derivation of formula (5.7) in the case of the last function, we find

$$\frac{\partial \tilde{p}}{\partial x} = -\frac{1}{\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty}\frac{\partial^2 \tilde{A}}{\partial \xi^2}\frac{d\xi}{\xi - \zeta} \tag{5.9}$$

Substituting expression (5.9) into the right-hand side of Eq. (2.31), we derive the Benjamin–Ono equation for the function $\tilde{A}$

$$(\tilde{A}-c)\frac{\partial\tilde{A}}{\partial x}=\frac{1}{\pi\sqrt{c-K_\infty}}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{\partial^2\tilde{A}}{\partial\xi^2}\frac{d\xi}{\xi-x+ct} \tag{5.10}$$

Hence, it has been shown that the system of equations (2.30)–(2.32), for solutions which depend on the combination $x-ct$, reduces to Eqs. (5.4) or (5.10). The possibility of such a reduction is based on the existence of the local relation (5.3) or the integral relations (5.7) and (5.8), which are special cases of the general relation (3.21) involving integral equation (3.22).

6. THE SIMPLEST EXACT SOLUTIONS FOR $c > K_\infty$

The single soliton solution of Eq. (5.10) is given by the expression

$$A=\frac{4c}{1+c^2(c-K_\infty)(x-ct)^2} \tag{6.1}$$

and exists when $K_\infty < c < 0$. If the amplitude of the soliton (6.1) is uniquely defined by its velocity

$$|A(x,t)|_{\max}=4|c|$$

then its characteristic width for a fixed amplitude can have any value. In particular, the "mass" of the soliton of the Eq. (6.1)

$$\mathcal{M}=\int_{-\infty}^{\infty}\frac{4|c|\,dx}{1+c^2(c-K_\infty)(x-ct)^2}=\frac{4}{(c-K_\infty)^{1/2}}\int_{-\infty}^{\infty}\frac{d\hat{x}}{1+\hat{x}^2}=\frac{4\pi}{(c-K_\infty)^{1/2}}$$

is not invariant in the whole set of solutions of the solitary wave type being a function of both c as well as of K_∞. In (6.1), $|c| < |K_\infty|$ everywhere, in which case $\mathcal{M}\to\infty$ if $|K_\infty|\to|c|+0$ and $\mathcal{M}\to 0$ if $|K_\infty|\to\infty$.

A periodic solution of Eq. (5.10) can be represented in the form

$$A=c+\frac{k}{(c-K_\infty)^{1/2}}\left[\frac{1+\sigma^2}{1-\sigma^2}-\frac{2(1-\sigma^2)}{1+\delta^2-2\sigma\cos[k(x-ct)]}\right] \tag{6.2}$$

The family of solutions (6.2) depends on the four parameters k, c, σ, K_∞ and exists when $0<\sigma<1$, $c>K_\infty$, $k>0$. The phase velocity c in (6.2) can be of either sign (unlike in (6.1)).

The invariance of Eq. (3.22) with respect to the transformation

$$t\to\mathcal{B}^{5/3}t,\qquad x\to\mathcal{B}x,\qquad A\to\mathcal{B}^{-2/3}A,\qquad K_\infty\to\mathcal{B}^{-2/3}K_\infty \tag{6.3}$$

and the transformation

$$x\to x-\mathcal{D}t,\qquad A\to A+\mathcal{D},\qquad K_\infty\to K_\infty-\mathcal{D} \tag{6.4}$$

enables one, without loss in generality in the class of solutions of the travelling wave type (6.2), to eliminate the time from Eq. (3.22) (that is, to change to a system of coordinates moving at velocity c by putting $\mathcal{D}=-c$), and also to consider spatially 2π–periodic solutions by selecting $\mathcal{B}=k^{-1}$ in (6.3). Having eliminated two parameters in the family of solutions (6.2) in this way, any solution from the given four-parameter family can be obtained from a narrower class by the application of transformations (6.3) and (6.4). Equation (3.22) is not invariant under a Galilean transformation since, according to (6.4), together with the transformation of the dependent and independent variables, the parameter K_∞ occurring in the equation also changes.

Thus, in Eq. (6.2), we put $c=0$, $k=1$ and specify that

$$\frac{1+\sigma^2}{1-\sigma^2}=\frac{(-K_\infty)^{1/2}}{\nu}$$

Then

$$A = \frac{1}{\nu} - \frac{2\nu}{|K_\infty|}\left\{1 - \left[1 - \frac{\nu^2}{|K_\infty|}\right]^{1/2} \cos x\right\}^{-1} \tag{6.5}$$

Here, $K_\infty < 0$, $\nu > 0$ and the two parameters play a different role: K_∞ occurs in (3.2) and actually changes the initial equation while ν determines the set of solutions when the equation is given.

While dwelling on the question of the different characteristic forms of the periodic solution, it is convenient to rewrite Eq. (6.5) in the form

$$A = \frac{1}{\nu}\left[1 - \frac{2\Omega}{1-(1-\Omega)^{1/2}\cos x}\right], \quad \Omega = \frac{\nu^2}{|K_\infty|}, \qquad 0 < \Omega < 1 \tag{6.6}$$

A graph of the function (6.6) is shown in Fig. 2 for $\nu = 4$, $\Omega = 0.99$ (the dashed curve) and $\nu = 4$, $\Omega = 0.01$ (the solid curve).

7. PROPAGATION OF THE WAVE FRONT FOR $c < K_\infty$

The possibility of representing Eq. (3.22) in the form (5.4) enables us to state at once the existence of the exact solution.

$$A = -|c|\left\{1 + \operatorname{th}\left[\frac{|c|}{2}(K_\infty + |c|)^{1/2}(x + |c|t)\right]\right\} \tag{7.1}$$

The wave (7.1) which is travelling upstream on a zero background provides a mechanism for the transfer of the perturbations from a downstream domain

$$A \to 0\ (x \to -\infty), \quad A \to -2|c|\ (x \to +\infty)$$

The inequality

$$c < \min\,[0, K_\infty] \tag{7.2}$$

serves as a condition for the existence of a solution of the form (7.1)

Inequality (7.2) permits the existence of solutions of the form (7.1) even under subcritical conditions, that is, when

$$-|c| < K_\infty < 0$$

Since the function (7.1) corresponds to a boundary-layer separation wave in supersonic flow [26], the possibility of solutions of a similar type at Mach numbers smaller than the critical Mach number ($K_\infty < 0$) reflects the specific nature of the class of transonic flows being considered: a whole series of qualitative features of separated (or pre-separated) conditions which are characteristic of a flow pattern outside the transonic range of velocities and substantially different for $M_\infty < 1$ and $M_\infty > 1$ are present in the proposed asymptotic model.

Together with (7.1), it can be shown that there are waves which propagate downstream, which are exact solutions of Eq. (3.22). They are described by a solution which differs from (7.1) in the replacement

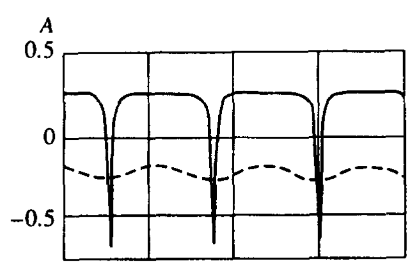

Fig. 2

of $|c|$ by $-|c|$. In this case, the wave front asymptotically joins the two steady conditions upstream and downstream:

$$A \to +2|c| \quad (x \to -\infty), \quad A \to 0 \quad (x \to +\infty)$$

These solutions exist subject to the condition $K_\infty > |c|$, that is, under supercritical transonic flow conditions.

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